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Some Suns in L_1

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INTRODUCTION

In [1] we studied a certain class of "one parameter" Chebychev sets in normed spaces (viz. the *EU-regular Chebychev sets* defined below) and obtained necessary and sufficient conditions for these sets to be suns. We then showed that if T is any infinite compact metric space, then $C(T)$ contains an (*EU*-regular) Chebychev set that is not a sun. Dunham [4] had given the first example of a Chebychev set in a normed space that is not a sun. His example is a "one-parameter" family in $C[0, 1]$. (Such an example is implicit in an earlier paper of Dunham [3].)

In this paper it is our purpose to show that the situation in the spaces of type L_1 is quite different. Indeed, *every EU-regular Chebychev set in an* L_1 *space is a sun.*

We give some definitions. A set B in a normed space X is *proximinal* if every point of X has at least one best approximation in B. A set B in X is *Chebychev* if every point in X has exactly one best approximation in B. A set B in X is a *sun* if, whenever $x \in X$ and b in B is a best approximation to x, b is also a best approximation to $b + \lambda(x - b)$ for all $\lambda \geqslant 0$.

Let I denote an interval of one of the following forms: $(-\infty, a]$, $[a, b]$, $[a, \infty)$, or $(-\infty, \infty)$. Let $M = \{F_c : c \in I\}$ be a curve contained in a normed space X. Then M is *E-regular* if: (i) $c_n \in I$ and $|c_n| \to \infty$ implies $\|F_{c_n}\| \to \infty$; (ii) for any $x \in X$, $c \in I$, $c_n \in I$ with $c_n \to c$,

$$\|F_c - x\| \leqslant \limsup \|F_{c_n} - x\|.$$

Also, M is *U-regular* if: (iii) $x^*(F_c)$ is a monotone function of c for every $x^* \in \operatorname{ext} S(X^*)$ (i.e., the set of extreme points of the unit ball, $S(X^*)$, of X^*); (iv) the linear span of $F_{c_1} - F_{c_2}$ is Chebychev whenever c_1, c_2 are in I. Finally, M is *EU-regular* if it is both E-regular and U-regular.

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In [1, Proposition 1.3], it was proved that the curve M is proximinal if it is E-regular and is Chebychev if it is EU-regular and if the function $c \to F_c$ is one-to-one. It was also proved [1, Theorem 1.17] that an EU-regular Chebychev curve is a sun if and only if the function $c \to F_c$ is continuous.

The Theorem

In the sequel, $(S, \mathscr{S}, \mu)$ will denote a measure space and $L_1 = L_1(S, \mathscr{S}, \mu)$ will denote the space of all (equivalence classes of) integrable functions x on S with the norm

$$\| x \| = \int_S | x | \, d\mu.$$

We assume further that $L_1{}^* = L_\infty(S, \mathscr{S}, \mu)$. (This will be the case, e.g., if $(S, \mathscr{S}, \mu)$ is σ-finite.) A set $A \in \mathscr{S}$ is called an *atom* if $0 < \mu(A) < \infty$, and whenever $B \in \mathscr{S}$, $B \subset A$, it follows that $\mu(B) = 0$ or $\mu(B) = \mu(A)$. Note that the assumption that $L_1{}^* = L_\infty$ implies that any union of atoms is measurable. Recall that

$$\operatorname{ext} S(L_1{}^*) = \{\sigma \in L_\infty : | \sigma | = 1 \text{ a.e. } \mu\}$$

LEMMA 1. *Let $h \in L_1 \backslash \{0\}$, let $\mathcal{A}$ denote the union of all atoms in $\mathscr{S}$, and let $\delta = \int_{S \backslash \mathcal{A}} | h | \, d\mu$. Then for each $\lambda \in [0, 1]$ there is a function σ on $S \backslash \mathcal{A}$ such that $| \sigma | = 1$ and*

$$\int_{S \backslash \mathcal{A}} h\sigma \, d\mu = \lambda\delta + (1 - \lambda)(-\delta).$$

Proof. Let

$$\nu(E) = \int_E | h | \, d\mu, \qquad E \subset S \backslash \mathcal{A}.$$

Then ν is a finite nonatomic measure, $\nu(\phi) = 0$, and $\nu(S \backslash \mathcal{A}) = \delta$. By a theorem of Liapunov [6] (see also [7]), the set $\{\nu(E) : E \subset S \backslash \mathcal{A}\}$ is convex so, for each $\lambda \in [0, 1]$, there exists $E \subset S \backslash \mathcal{A}$ such that $\nu(E) = \lambda\delta$. Thus, setting

$$\begin{aligned} \sigma &= \operatorname{sgn} h \text{ on } E \\ &= -\operatorname{sgn} h \text{ on } (S \backslash \mathcal{A}) \backslash E, \end{aligned}$$

it follows that

$$\int_{S \backslash \mathcal{A}} h\sigma \, d\mu = \int_E | h | \, d\mu - \int_{(S \backslash \mathcal{A}) \backslash E} | h | \, d\mu = \lambda\delta - (1 - \lambda)\, \delta. \quad \blacksquare$$

LEMMA 2. *Let $h \in L_1 \backslash \{0\}$: The following statements are equivalent.*

(1) span$\{h\}$ *is Chebychev*

(2) span$\{h\}$ *is "interpolating," i.e., there does not exist* $x^* \in \operatorname{ext} S(L_1{}^*)$ *with* $x^*(h) = 0$

(3) $\mathscr{S}$ *contains atoms and*

$$\int_{S\backslash\mathcal{A}} |h| \, d\mu < \min \left\{ \left| \int_{\mathcal{A}} h\sigma \, d\mu \right| : \sigma \in L_\infty, |\sigma| = 1 \right\};$$

where $\mathcal{A}$ *is the union of all atoms in S.*

Proof. The equivalence of (1) and (2) is a consequence of some results of Phelps [8, Theorem 1.8 and Lemma 2.4].

(2) $\Rightarrow$ (3). If span$\{h\}$ is Chebychev, then by a result of Dye [8, Theorem 2.5] $\mathscr{S}$ must have atoms.

Let $\mathcal{A}$ denote the union of all atoms in S, $\delta = \int_{S\backslash\mathcal{A}} |h| \, d\mu$, and

$$r = \min \left\{ \left| \int_{\mathcal{A}} h\sigma \, d\mu \right| : \sigma \in L_\infty, |\sigma| = 1 \right\}.$$

Suppose $r \leqslant \delta$. Then by Lemma 1, there is a function σ on $S\backslash\mathcal{A}$ such that $|\sigma| = 1$ and

$$\int_{S\backslash\mathcal{A}} h\sigma \, d\mu = r.$$

Define σ on $\mathcal{A}$ so that $|\sigma| = 1$ and

$$\int_{\mathcal{A}} h\sigma \, d\mu = -r.$$

Then $\int_S h\sigma \, d\mu = 0$, and this contradicts the fact that span$\{h\}$ is interpolating. Hence, $\delta < r$, and (3) holds.

(3) $\Rightarrow$ (2). Assume (3) holds but (2) fails. Then there is a function β with $|\beta| = 1$ such that $\int_S h\beta \, d\mu = 0$. In particular,

$$\int_{S\backslash\mathcal{A}} h\beta \, d\mu = - \int_{\mathcal{A}} h\beta \, d\mu.$$

Thus,

$$\left| \int_{\mathcal{A}} h\beta \, d\mu \right| = \left| \int_{S\backslash\mathcal{A}} h\beta \, d\mu \right| \leqslant \int_{S\backslash\mathcal{A}} |h| \, d\mu,$$

which contradicts (3). ∎

One interesting consequence of this result (although it is not pertinent to the present work) is that *if* span$\{h\}$ *is Chebychev in* $L_1(S, \mathscr{S}, \mu)$, *then* span$\{h|_{\mathcal{O}}\}$ *is Chebychev in* $L_1(\mathcal{O}, \mathscr{S} \cap \mathcal{O}, \mu|_{\mathcal{O}})$, *where* $\mathcal{O}$ *is the union of all atoms in S.*

LEMMA 3. *Let M be a U-regular set in* L_1. *Then any weakly convergent sequence from M converges in norm.*

Proof. If S is purely atomic, the result is a consequence of the well-known fact that a sequence that converges weakly must actually converge in norm (see, e.g., [2, p. 295]). Let $F_{c_n} \in M$ and $F_{c_n} \to y$ weakly. Given any subsequence (n_i) of the natural numbers, let

$$h_i = F_{c_{n_i}} - F_{c_{n_{i+1}}}.$$

Then span$\{h_i\}$ is a Chebychev subspace for each i. By Lemma 2

$$(*)\quad \int_{S\setminus\mathcal{O}} |h_i|\, d\mu \leqslant \int_{\mathcal{O}} |h_i|\, d\mu, \qquad \text{for each } i,$$

where $\mathcal{O}$ is the union of atoms in S. Since $h_i \to 0$ weakly, it follows that $h_i|_{\mathcal{O}} \to 0$ weakly (regarded as elements of $L_1(\mathcal{O}, \mathscr{S} \cap \mathcal{O}, \mu|_{\mathcal{O}})$). By the result stated at the beginning of the proof, $\int_{\mathcal{O}} |h_i|\, d\mu \to 0$. From the inequality (*), $\int_{S\setminus\mathcal{O}} |h_i|\, d\mu \to 0$, and hence, $\|h_i\| = \int_S |h_i|\, d\mu \to 0$. That is,

$$\|F_{c_{n_i}} - F_{c_{n_{i+1}}}\| \to 0.$$

Since the subsequence (n_i) was arbitrary, (F_{c_n}) must be a Cauchy sequence, and hence, converges. ∎

THEOREM. *Every EU-regular Chebychev set in* L_1 *is a sun.*

Proof. Let $M = \{F_c : c \in I\}$ be an EU-regular Chebychev set. If M is not a sun, then by [1, Theorem 1.17], the map $F_{(\cdot)}$ is discontinuous. Thus, there is a $c_0 \in I$, a sequence (c_n) in I converging monotonically to c_0, and an $\epsilon > 0$ such that

$$(*)\quad \|F_{c_n} - F_{c_0}\| \geqslant \epsilon, \qquad \text{for every } n.$$

We may assume $c_n \downarrow c_0$. For each $x^* \in \operatorname{ext} S(L_1^*)$, $x^*(F_{c_n})$ is a bounded monotone sequence, and hence, converges. It follows that $x^*(F_{c_n})$ converges for every $x^* \in \operatorname{co}(\operatorname{ext} S(L_1^*))$. Now L_1^* is isometric with L_∞, which is isometric with $C(T)$, for some compact extremally disconnected Hausdorff space T [2, p. 445.]. Hence, by a well-known result of Goodner [5], the unit ball of $C(T)$, $S(C(T))$, is the norm closed convex hull of its extreme points. Therefore,

$$S(L_1^*) = \overline{\text{co}}(\text{ext } S(L_1^*)).$$

Let $K = \sup_n \| F_{c_n} \|$.
Then $K < \infty$ by E-regularity. It follows by a standard argument that for every $x^* \in L_1^*$, $x^*(F_{c_k})$ converges. Thus, (F_{c_k}) is a weak Cauchy sequence. Since L_1 is weakly complete [2, p. 290], (F_{c_k}) converges weakly to some element of L_1. By Lemma 3 and the fact that M is closed, $F_{c_k} \to F_c$ for some $c \in I$. By [1, Lemma 1.9], an E-regular curve has a closed graph and so $F_c = F_{c_0}$. But this contradicts (*). ∎

PROBLEM. More generally, must every Chebychev curve in L_1 be a sun?

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